OSCILLATING-CURRENT CIRCUITS

AN EXTENSION OF THE THEORY OF GENERALIZED ANGULAR VELOCITIES, WITH APPLICATIONS TO THE COUPLED CIRCUIT AND THE ARTIFICIAL TRANSMISSION LINE

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V. BUSH

ABSTRACT

OF

A THESIS

SUBMITTED TO THE FACULTY OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY IN PART FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF ENGINEERING

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CONTENTS.

LIST OF SYMBOLS EMPLOYED IN THESIS.

i The instantaneous oscillating current in a branch of a network—amperes

n A generalized angular velocity of oscillation—hyperbolic radians per second $\angle$

Z Generalized impedance—ohms $\angle$

E Initial potential—volts

ϵ Naperian base—2.718 . . . .

$n_1 n_2$ Roots of the equation $Z=0$, hyperbolic radians per second $\angle$

C Total capacitance—farads

L Total self inductance—henries

M Total mutual inductance—henries

R Total resistance—ohms

Constants of the primary and secondary of a coupled circuit are distinguished by subscripts

$\alpha, \beta, \gamma, \delta, \eta$ Coefficients of the equation $Z=0$ for the coupled circuit

q Correction to be applied to the absolute values of the free angular velocities of a resistanceless coupled circuit to obtain the absolute values of the angular velocities of the complete circuit. —— numeric

p A correction to be added and subtracted to $\frac{\alpha}{4}$ to obtain the decrements of the complete coupled circuit. —— hyp. rad. per sec.

s, t Sum and difference respectively of the squares of the angular velocities of the resistanceless coupled circuit. $\left(\frac{\text{hyp. rad.}}{\text{sec.}}\right)^2$

j The pure imaginary, $\sqrt{-1}$

A A generalized amplitude of current oscillation—amperes $\angle$

m Number of sections of an artificial line

h Auxiliary constant numeric

$\angle$ This sign appended to the units of an equation denotes that the expression contains, in general, complex quantities

OSCILLATING–CURRENT CIRCUITS.

INTRODUCTION.

Heaviside,* and since then several others,† have shown that for the free oscillations of a network the generalized impedance, formed from the constants of the network and the complex angular velocity of oscillation, is zero for any complete circuit. This principle enables the frequencies and decrements of the free oscillations of a network to be readily found. There is a similar principle which enables the finding of the amplitudes of free oscillation at the several frequencies, which is also in Heaviside, derived from a series of theorems concerning the distribution of energy during subsidence. It is the purpose of the thesis, of which this is an abstract, to demonstrate the application of this second principle to practical engineering problems.

The principle may be stated as follows: If Z is the generalized impedance of a branch of the network initially containing a store of energy, corresponding to the initial voltage E, and if n is the complex angular velocity of oscillation, so that $Z = f(n)$, then the first order term in the Taylor expansion of Z, namely, $n \dfrac{dZ}{dn}$, will be of the nature of an impulsive impedance; and the oscillatory current will be of the form:

$$i = \sum \frac{E}{n\dfrac{dZ}{dn}} \epsilon^{nt} \qquad \text{amperes } \angle$$

where the summation extends over the roots $n_1, n_2, --$ of the equation $Z = 0$.

It will be convenient to call the expression $n \dfrac{dZ}{dn}$ the "threshold impedance."

The equation, as given, applies to the current in the branch initially charged, where the generalized and threshold impedances are formed for that branch.

The discussion of the application of this principle to various typical networks has indicated the truth of the following additional propositions which will be found useful in attacking particular problems:

* Heaviside. Electrical Papers, Electromagnetic Theory, Vol. II.
† Campbell, Proc. AIEE, 1911;
Kennelly, Proc. IRE, 1915;
Eccles & Makower, Electrician, 1915.

(1) In determining the amplitude of oscillation at some point of the network distant from the branch initially charged, the generalized impedances of the elements combine in the manner of simple resistances. Upon combining with the generalized impedance of an element, each term of a current or voltage expression is combined with the generalized impedance of the element formed for the free angular velocity of the term considered.

(2) When several stores of energy are simultaneously discharged they may be considered separately and the results added.

(3) In order to ensure that the correct free angular velocities be obtained, the generalized impedance should be formed for the branch under examination; as in special cases certain free angular velocities may be absent in particular branches of the network.

(4) The threshold impedance is formed always from the generalized impedance which considers the initially charged element as the main branch.

(5) The sudden application of a steady electromotive force may be treated as the inverse of the discharge from the final state attained.

(6) The sudden application of an alternating electromotive force may be treated in similar manner, the unbalanced stores of energy being in this case the differences between the initial stores of energy in the network, and the energies at the same points of the network corresponding in the steady state to the point of the voltage wave at which it was suddenly applied.

The method of applying the threshold impedance is shown by various examples. One of these, the series circuit containing resistance, inductance, and capacitance is included here for illustration.

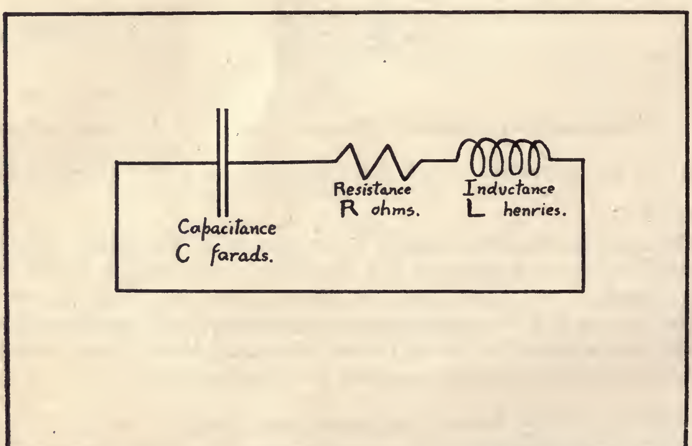

Fig. 1. Simple Series Oscillating Circuit.

In this circuit (see fig. 1) the generalized impedance will be:

$$Z = R + Ln + \frac{1}{Cn} \qquad \text{ohms } \angle$$

Equating to zero and solving for n, we obtain the free angular velocities:

$$n_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \frac{\text{hyp. rad.}}{\text{sec.}} \angle$$

$$n_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \frac{\text{hyp. rad.}}{\text{sec.}} \angle$$

The threshold impedance is:

$$n\frac{dZ}{dn} = Ln - \frac{1}{Cn} \qquad \text{ohms } \angle$$

If now we consider the condenser as discharging through the circuit from an initial voltage E, the current will be:

$$i = \sum_{n=n_1}^{n=n_2} \frac{E}{n\frac{dZ}{dn}} \epsilon^{nt} \qquad \text{amperes } \angle$$

or

$$i = \frac{E}{Ln_1 - \frac{1}{Cn_1}} \epsilon^{n_1 t} + \frac{E}{Ln_2 - \frac{1}{Cn_2}} \epsilon^{n_2 t} \qquad \text{amperes } \angle$$

which, with the values of n_1 and n_2 given above, is the complete oscillatory solution. This expression may be reduced to the usual form by inserting the values of n_1 and n_2. There will, of course, be three cases according as the quantity under the radical is positive, zero, or negative. For the third case the expression becomes upon reducing:

$$i = \frac{E}{L\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \epsilon^{-\frac{Rt}{2L}} \sin \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\, t \qquad \text{amperes}$$

which is the solution obtained by the usual methods. The solutions for the other cases may be obtained by similar reductions.

It will be noted that this method of solving the circuit is much more concise and direct than is the method of determining the constants of integration in the differential equation solution, in accordance with the boundary conditions. It is also convenient to retain all three cases in the single expression.

If we wish the oscillatory voltage across, for instance, the reactor in this circuit, we may obtain it by multiplying the oscillatory current by the generalized impedance of the reactor, and treat the current terms separately, thus:

$$e_L = Lni = \frac{ELn_1}{Ln_1 - \frac{1}{Cn_1}} \epsilon^{n_1 t} + \frac{ELn_2}{Ln_2 - \frac{1}{Cn_2}} \epsilon^{n_2 t} \qquad \text{volts } \angle$$

and this expression may also be reduced by inserting the values of n_1 and n_2.

THE COUPLED CIRCUIT.

The coupled circuit has been thoroughly solved by the method of differential equations.* These solutions have been discussed from the point of view of the applications of this circuit, particularly to radio work. Many approximate solutions have been obtained for the case of the free discharge of the primary condenser, either by neglecting the effects of resistance, or the reaction of the secondary upon the primary, or in some similar way. The complete exact solution has been generally avoided, principally because of its complication. The resistance operator method, or the method of generalized angular velocities, has also been applied to this circuit in as far as the frequencies and decrements are concerned.† This method gives the same equation for the determination of the free angular velocities as does the differential equation solution, namely:

$$C_1C_2(L_1L_2 - M^2)n^4 + C_1C_2(R_1L_2 + R_2L_1)n^3 + (C_1L_1 + C_2L_2 + C_1C_2L_1L_2)n^2 + (C_1R_1 + C_2R_2)n + 1 = 0 \qquad \text{numeric } \angle$$

where n is the complex angular velocity, and the constants are those shown on fig. 2.

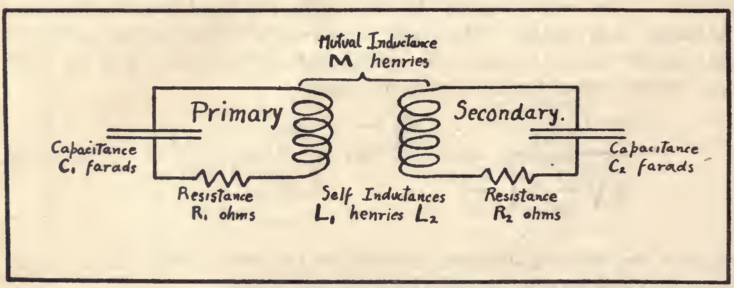

Fig. 2. Inductively Coupled Circuit.

* Bjerkness, Wied. Ann. 55, 1895; Oberbeck, Wied. Ann. 55, 1895; Domalys & Kolacek, Wied. Ann. 57, 1896; Wien, Wied. Ann. 61, 1897; Rayleigh, Theory of Sound; Braun, Phys. Zs. 3, 1901; Drude, Ann. d. Phys. 13, 1904; Jones, Phil. Mag. 1907; Cohen, Bul. Bu. Stds. 5, 1909; Pierce, Proc. Am. Ac. A. & Sc. 46, 1911; Fleming, Proc. Phys. Soc. 1913.

† Eccles, Phys. Soc. Proc. 24, 1912. Kennelly, Proc. IRE, 1915.

The solution of this fourth degree equation is laborious, and may be avoided in the following manner. The equation may be written in the form:

$$n^4+\alpha n^3+\beta n^2+\gamma n+\delta=0 \qquad \left(\frac{\text{hyp. rad.}}{\text{sec.}}\right)^4 \angle$$

and if we treat the same circuit without resistance we obtain the easily solved equation:

$$n^4+\eta n^2+\delta=0 \qquad \left(\frac{\text{hyp. rad.}}{\text{sec.}}\right)^4 \angle$$

The roots of this last equation will differ but little in absolute value from the absolute values of the roots of the complete equation. If $(1+q)$ and $(1-q)$ are the correction factors to be applied to the absolute values of the resistanceless roots in order to obtain the absolute values of the complete roots, we may find an expression for q by means of the algebraic relations between the roots and coefficients of the above equations; and in deriving this relation the square of q may be neglected. This expression is:

$$q=\frac{\alpha\gamma s+st^2-\beta t^2-\gamma^2-\alpha^2\delta}{2\,t(t^2+\alpha\gamma+2\,s^2-2\,s\beta)} \qquad \text{numeric}$$

where s is the sum and t the difference of the squares of the roots of the resistanceless equation.

In a similar manner the relation may be derived:

$$p=\frac{2\,\gamma-\alpha s+\alpha qt}{4\,t-8\,qs} \qquad \frac{\text{hyp. rad.}}{\text{sec.}}$$

where $\left(\frac{\alpha}{4}+p\right)$ and $\left(\frac{\alpha}{4}-p\right)$ are the decrements in the solution of the complete circuit. In this manner the frequencies and decrements of the oscillations in the coupled circuit may be obtained without the necessity of solving the fourth degree equation.

This method, tested on a typical circuit with constants:

$C_1=10^{-9}$ farads
$C_2=10^{-10}$ farads
$R_1=1000$ ohms
$R_2=2000$ ohms
$L_1=0.025$ henries
$L_2=0.040$ henries
$M=0.020$ henries

gave by exact solution:

$$n=\begin{cases}-57361.1\pm j\,664749.\\-17638.9\pm j\,192683.\end{cases}$$

and by the approximate method:

$$n=\begin{cases} -57361.0\pm j\,664750. \\ -17639.0\pm j\,192680. \end{cases}$$

The amplitudes of oscillation may be readily found by the use of the threshold impedance. If we consider the discharge of the primary condenser, so that the primary is the main branch, the threshold impedance is:

$$n\frac{dZ}{dn}=L_1n-\frac{1}{C_1n}-M^2\frac{L_2n^3+2R_2n^2+\frac{3n}{C_2}}{\left(R_2+L_2n+\frac{1}{C_2n}\right)^2} \qquad \text{ohms } \angle$$

Inserting into this expression the four roots n_1, n_2, n_3, n_4 of the equation $Z=0$, gives four particular values of the threshold impedance. Dividing the initial primary condenser voltage by each of these values gives the four complex amplitudes for the primary current expression:

$$i_1=A_1\epsilon^{n_1t}+A_2\epsilon^{n_2t}+A_3\epsilon^{n_3t}+A_4\epsilon^{n_4t}, \qquad \text{amperes } \angle$$

This expression may be readily reduced to trigonometric form, when the imaginary portions of the expression cancel out.

An examination of the generalized impedance of the several elements of the circuit gives for the ratio between the primary and secondary amplitudes:

$$-\frac{Mn}{R_2+L_2n+\frac{1}{C_2n}} \qquad \text{numeric } \angle$$

The four values of this ratio applied to the primary amplitudes give the corresponding secondary amplitudes.

The results above were checked by means of oscillograms taken upon a typical coupled circuit. The constants chosen

$R_1=1.937$ ohms
$R_2=2.531$ ohms
$L_1=7.52 \times 10^{-3}$ henries
$L_2=7.63 \times 10^{-3}$ henries
$M=3.475 \times 10^{-3}$ henries
$C_1=13.51$ microfarads
$C_2=24.62$ microfarads

gave frequencies of oscillation 609.5 ∾ and 339.2 ∾ which were within convenient range for the oscillograph. The computed points checked the oscillograms within the errors of measurement. A solution was also made by differential equations as a check.

APPLICATION TO THE ARTIFICIAL LINE.

There has been much difficulty encountered in the analysis of smooth line transients. The cable has been comparatively easily handled,* but the analysis of the aerial line has given in general results too complicated for engineering use. For experimental analysis for steady state phenomena the lumped artificial line has proved invaluable,† but there has been much doubt as to just how far such a line of a given number of sections could be trusted for transient effects.‡

The method of generalized angular velocities is applied in the thesis to the analysis of the oscillations of the artificial line under certain typical conditions. The distant-end current on a grounded artificial line, when a steady voltage is suddenly applied at the home end, is considered for the artificial cable, and the artificial aerial line. The π line is used, but the formulas apply also to the T line with small changes.

The purpose of this analysis is to determine, for specific cases, the number of sections requisite in an artificial line, in order that it may represent its corresponding smooth line, not only for the steady state, but also for certain transient effects, to a sufficient degree of approximation for engineering investigations.

The method used is simply to analyze artificial lines with various numbers of sections, considered simply as networks with concentrated constants. The results of these successive solutions are grouped, and from them is derived the solution for the general case of m sections.

The general solutions for the application of a steady electromotive force to the grounded artificial line obtained in this way follow:

For the cable of m sections containing resistance and capacitance only:

$$i_B = \frac{E}{R}\left[1 + \frac{h_{m-1}}{2}\epsilon^{\frac{m^2 h_1 t}{RC}} - \frac{h_{m-2}}{2}\epsilon^{\frac{m^2 h_2 t}{RC}} + \ldots\right] \quad \text{amperes}$$

where

i_B is the received current
E the applied steady voltage
R the total resistance
C the total capacitance.

* Kelvin, Proc. Roy. Soc. 1855;
Poincaré, Ec. Elect. 40, 1904;
Malcolm, Electrician 1911; 12.

† Pupin, Trans. AIEE 1890, 1900; Trans. Am. Math. Soc. 1900;
Kennelly, Proc. Am. Acad. Arts & Sci., 44, 1908;
Huxley, Thesis M. I. T., 1914.

‡ Cunningham and Davis, Proc. AIEE 1911, 1912;
Ricker, Thesis M. I. T. 1915.

The values of h are found as roots of the auxiliary equation:

$$(h+2)^{m-1}-(m-2)(h+2)^{m-3}+\frac{(m-3)(m-4)}{2\,!}(h+2)^{m-5}- \ldots = 0$$

numeric

where there are $\frac{m}{2}$ terms if m is even, and $\frac{m+1}{2}$ terms if m is odd.

A curve for obtaining these roots is presented for convenient use in practical cases.

For the aerial line containing resistance, capacitance and inductance, the corresponding equation is:

$$i_B=\frac{E}{R}\left[1-\epsilon^{-\frac{Rt}{L}}+\frac{h_{m-1}\frac{R}{2L}}{\sqrt{-\frac{m^2h_1}{LC}-\left(\frac{R}{2L}\right)^2}}\epsilon^{-\frac{Rt}{2L}}\sin\sqrt{-\frac{m^2h_1}{LC}-\left(\frac{R}{2L}\right)^2}\,t\right.$$

$$\left.-\frac{h_{m-2}\frac{R}{2L}}{\sqrt{-\frac{m^2h_2}{LC}-\left(\frac{R}{2L}\right)^2}}\epsilon^{-\frac{Rt}{2L}}\sin\sqrt{-\frac{m^2h_2}{LC}-\left(\frac{R}{2L}\right)^2}\,t+\ldots\right]$$

amperes

As the line is subdivided an oscillatory term appears in this equation for each section introduced.

This aerial line formula was checked by means of oscillograms taken upon a typical artificial aerial power transmission line at Pierce Hall, Harvard University.* Twelve sections were used, representing a line of the following constants:

#000 A.W.G. aluminum stranded conductors
Overstrand diameter 0.47 inches
Interaxial distance 90.5 inches
Length 596.4 miles.
Total inductance 1.035 henries
Total capacitance 8.10×10^{-6} farads
Total resistance 300.1 ohms

The elements of this artificial line were grouped in such a manner that it was arranged as a π line of various numbers of sections. The arrival curve of current computed and plotted to the scale of the oscillograms showed a check to a reasonable degree of accuracy.

* Kennelly and Tabossi, Elec. World 1912.

In order to determine the relation between the artificial and smooth cables, the cable formula was plotted for the numerical case:

$$E = 200 \text{ ohms}$$
$$C = 10^{-6} \text{ farads}$$
$$R = 20 \text{ volts}$$

for various numbers of sections. It was found that with these constants, the arrival curve on a six section artificial cable coincided to a sufficient degree of accuracy, for the purposes of engineering, with the smooth line arrival curve as plotted from Kelvin's formula:

$$i_B = \frac{E}{R}\left[1 - 2\epsilon^{-\frac{\pi^2 t}{RC}} + 2\epsilon^{-\frac{4\pi^2 t}{RC}} - \ldots\right] \quad \text{amperes}$$

The limiting value of the artificial aerial line formula as the number of sections is indefinitely increased was also considered. From the fact that the artificial cable formula approaches Kelvin's smooth line formula in the limit, were derived the limits of the various values of m^2h and h as $m \doteq \infty$. A certain approximation was also made because of the fact that in lines encountered in practice, $\left(\frac{R}{2L}\right)^2$ may be neglected in comparison with $\frac{m^2h}{LC}$. Applying these facts to the artificial aerial line formula gave the following expression for the received current on a grounded smooth aerial line when a steady voltage is suddenly applied at the home end:

$$i_B = \frac{E}{R}\left[1 - \epsilon^{-\frac{Rt}{L}}\right] - \frac{E}{\sqrt{\frac{L}{C}}}\epsilon^{-\frac{Rt}{2L}} F(t) \quad \text{amperes}$$

where $F(t)$ is the discontinuous function represented in fig. 3.

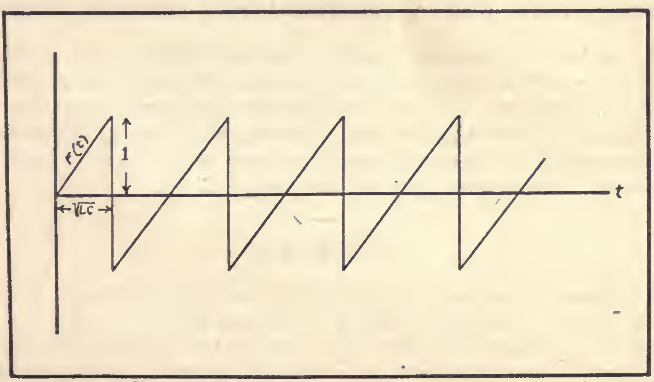

Fig. 3. The function F(t) in the aerial line formula.

The arrival curve plotted from this formula for the smooth line on which the oscillograms were taken is shown in fig. 4.

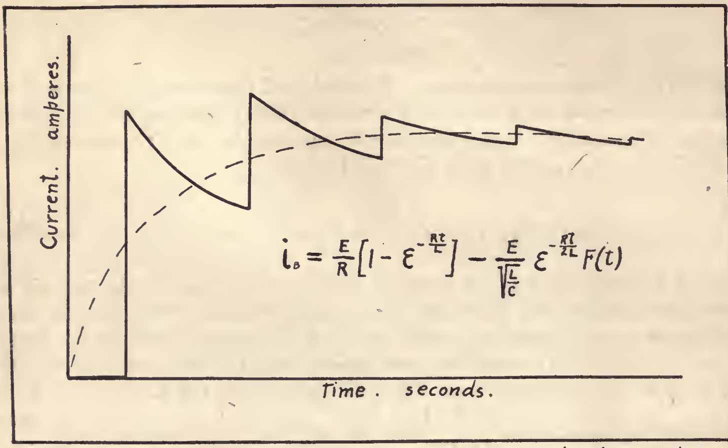

Fig. 4. *Arrival Current, Smooth Aerial Line.*

A comparison of this smooth line curve with the artificial line arrival curves showed that the artificial line of four sections, or less, did not well approximate the smooth line, for the transient due to the sudden application of a steady voltage. The artificial line of twelve sections approximated the smooth line fairly well; but a still greater number of sections would be necessary, in order to enable the artificial line to be used for experimental investigation with this type of transient.

SUGGESTION FOR A CONTINUATION OF THE WORK.

The method of generalized angular velocities, applied to the oscillations of networks with concentrated constants, has proved to be valuable for engineering purposes. It is believed that the same method may be profitably applied to networks containing branches with distributed constants. A starting point for such work would be found in Heaviside's application of the resistance operator to the smooth line.

SUMMARY.

In addition to the theorem which determines the free angular velocities of oscillation of a network, there is a theorem which will determine the amplitudes. This theorem involves a "threshold impedance" which may be formed for any circuit with concentrated constants, and which enables the amplitudes of oscillation to be found from the initial potential of the unbalanced energy.

An application of this method to the coupled circuit gives an easily applied and convenient complete solution for the primary condenser discharge.

Applied to the artificial line, it enables the lumped line of a given number of sections to be compared with the represented smooth line for certain transient effects.

The writer wishes to express his thanks to Prof. D. C. Jackson, Dr. A. E. Kennelly, and other members of the Department of Electrical Engineering who have assisted him in the preparation of the thesis.

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